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Fundamentals concerning wave loading on offshore structures

By JAMES LIGHTHILL
University College, Gower Street, London, WC1E 6BT, UK

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This article is aimed at relating a certain substantial body of established material concerning wave loading on offshore structures to fundamental principles of mechanics of solids and of fluids and to important results by G. I. Taylor (1928*a*, *b*). The object is to make some key parts within a rather specialised field accessible to the general fluid-mechanics reader.

The article is concerned primarily to develop the ideas which validate a separation of hydrodynamic loadings into vortex-flow forces and potential-flow forces; and to clarify, as Taylor (1928*b*) first did, the major role played by components of the potential-flow forces which are of the second order in the amplitude of ambient velocity fluctuations. Recent methods for calculating these forces have proved increasingly important for modes of motion of structures (such as tension-leg platforms) of very low natural frequency.

1. Introduction

G. I. Taylor was prone to lay stress on those basic principles which are common to solid mechanics and fluid mechanics. His work exemplified also how, by a proper concentration on fundamentals, the level of understanding needed to tackle important engineering problems could be greatly heightened. In that spirit I want to describe some current techniques for estimating wave loading on offshore structures from the standpoint of the fundamental mechanics of solids and fluids.

Following up an earlier, much more detailed survey (Lighthill 1979) I propose to outline the basic physical ideas underlying the 'Morison equation' approach to wave loading estimation. Then I want to argue that, as we necessarily move to more refined methods of estimation, we can appropriately continue to separate hydrodynamic loadings (as Morison's equation does) into vortex-flow forces and potential-flow forces. However, second-order terms in the interaction between the potential-flow component of fluid motion and the structure need to be taken into account (in a manner which Taylor (1928*b*) first pioneered), and these are particularly important for the compliant structures that are increasingly being employed as oil extraction programmes move into deeper and deeper water.

2. Basic principles concerning vorticity

The analysis of fluid loadings into vortex-flow forces and irrotational-flow forces is founded initially, of course, upon that analysis of the most general deformation of a small spherical element which is fundamental both to solid mechanics and fluid mechanics. Displacement of such a small spherical element, relative to its centre, is divided into

(*a*) pure straining motion, devoid of angular momentum; and

(*b*) rigid rotation with an angular velocity $\frac{1}{2}\boldsymbol{\omega}$, where $\boldsymbol{\omega}$ is the vorticity.

For a homogeneous fluid with viscous stresses neglected, a small spherical element is acted upon only by pressure forces directed through its mass centre so that the rate of change of the angular momentum in motion (*b*) is zero. This fact, combined with consideration of how motion (*a*) is altering the principal moments of inertia of the spherical element, was shown by Lighthill (1963) to yield a simple derivation of the familiar rules that, for such a fluid, vortex lines are convected by the fluid motion (Helmholtz's theorem) with vorticity being altered in magnitude and direction to the extent that such a convected vortex line is stretched and rotated. To those convection effects, viscous stresses add a diffusion of the vorticity with a diffusivity ν taking values between 1 and 2 $mm^2\ s^{-1}$ for the ocean. This implies that diffusion distances during a typical ocean-wave period are of the order of a centimetre or less.

Vorticity in a homogeneous fluid is created only at solid boundaries. For Reynolds numbers of engineering interest the rate of vorticity production at a stationary boundary is dominated (Lighthill 1963, p. 54) by a term whose magnitude is the tangential component of the pressure-gradient force per unit mass. Its direction, however, is at right-angles to that component so that, as might be expected from (*a*) and (*b*) above, the angular momentum of a spherical element of fluid tangential to the boundary (where it satisfies the no-slip condition) is increasing at the same rate as would the angular momentum of a non-slipping rigid sphere subjected to the same force per unit mass acting through its centre.

Lighthill (1963) showed how all of the above rules, for the rate of generation of vorticity at a solid boundary and for its subsequent convection and diffusion, can be used to achieve a sound understanding of the general properties of both unseparated and separated boundary layers. These ideas were later expounded by him in the context of hydrodynamic loadings on offshore structures (Lighthill 1979) and have recently been given a central place in an introductory textbook on theoretical fluid mechanics (Lighthill 1986).

3. Basic principles concerning irrotational flow

Alongside the crucially important properties of vorticity it is necessary to utilize also the properties of irrotational flows. In an irrotational flow, of course, the instantaneous motion of every spherical element of fluid takes the above form (*a*), devoid of angular momentum.

Motion started impulsively from rest is initially irrotational because any vorticity, necessarily created at the solid boundary, has not as yet had time to be moved away from it by diffusive and convective action (including any boundary-layer separation). That initial irrotational flow necessarily possesses a single-valued velocity potential ϕ (such that $-\rho\phi$, where ρ is the density, specifies the impulsive pressure acting in the fluid during the impulsive start); note that, in a multiply connected region of fluid, this property distinguishes it from other irrotational flows, which lack a single-valued velocity potential because the circulation around certain circuits is non-zero.

For motions of externally unbounded fluid around a solid structure, there is just one such potential flow satisfying appropriate boundary conditions; these conditions specify the fluid motion far from the structure while, at its surface, they equate the normal components of velocity of the structure and the fluid. This well known uniqueness theorem for potential flows is reminiscent of results in the mechanics of solids, where a deformation field may be determined uniquely from appropriate boundary conditions specifying forces or displacements. Also, just as that deformation field may be characterized as the unique field compatible with the boundary conditions which possesses minimum strain energy, so also the potential flow around a structure may be characterized as the unique motion compatible with the boundary conditions for which the disturbances to the far-field motion possess minimum kinetic energy.

Kelvin proved this Minimum Energy Theorem by making an analysis of a complete flow field in a form somewhat analogous to that given for a spherical element in §2. He analysed the motion of externally unbounded fluid as a linear combination of

(*a*) the potential flow that satisfies the boundary conditions; and

(*b*) a residual vortex motion satisfying zero boundary conditions.

Here, 'zero boundary conditions' imply zero fluid motion far from the structure as well as zero normal velocity at its surface. The analogy with the analysis in §2 is only partial, of course; because a spherical element, which in the potential flow does only undergo pure straining, is in general subject to both rotation and straining in the vortex motion.

Kelvin showed that the kinetic energy of the disturbances to the far-field motion is equal to the sum of its values for the potential flow (*a*) and the vortex motion (*b*); in fact, the 'cross terms' which might naturally be expected are proved to make a zero contribution by applying the Divergence Theorem to the product of the disturbance potential in (*a*) with the velocity field in (*b*). It follows that the kinetic energy is least when the motion (*b*) vanishes.

However, the Kelvin analysis into (*a*) and (*b*) above has a wide range of practical applications in addition to its theoretical use for proving the Minimum Energy Theorem. Necessarily, offshore structures shed vorticity in most substantial amounts from separated boundary layers; indeed, design options such as might be aimed at minimizing shed vorticity (including, in other engineering contexts, the use of 'streamlined' cross-sections) are unavailable for structures that must withstand fluid flow from any direction. Therefore, analysis of the flow around such structures into a potential flow and a residual motion associated with the shed vorticity may be most fruitful.

4. Energy approaches to Morison's equation

A preliminary idea of the spatial and temporal distribution of forces on elements of offshore structures may be derived from arguments making direct use of Kelvin's result that the disturbance energies for components (*a*) and (*b*) of an externally unbounded flow are additive. Here, following in summary the more detailed survey by Lighthill (1979), we may preface accounts of more versatile and quantitatively valuable methods by looking briefly at such energy approaches.

Consider first the translational motion of a body with variable velocity $U(t)$, in the negative x-direction, through externally unbounded fluid which is at rest far from the body. Then the irrotational part (*a*) of the fluid motion depends only upon those boundary conditions which it satisfies instantaneously. This is the part (unlike (*b*), the vortex motion) which is devoid of any 'memory' for earlier values taken by $U(t)$; rather, it is proportional simply to the current value of U, and its kinetic energy is proportional to U^2.

The coefficient of proportionality is usually written $\frac{1}{2}M_a$ so that the kinetic energy

$\frac{1}{2}M_a U^2$ of the potential flow due to the body's motion at velocity U can be thought of as if it were the kinetic energy of an added mass M_a of fluid which the body's motion effectively drags along with it. Then, if U is increasing as a function of the time t, the rate of increase of this kinetic energy in part (a) of the fluid motion can be written $M_a U\dot{U}$. To feed this increase the body must act upon the fluid with the classical thrust $M_a \dot{U}$ overcoming the equal and opposite potential-flow drag,

$$M_a \dot{U}, \tag{1}$$

of the fluid on the body.

Simultaneously, the kinetic energy of part (b), the vortex motion, is increasing as more and more vorticity is shed into the wake, where the vortex lines are subsequently convected and diffused. The rate of working by the thrust with which the body acts upon the fluid is necessarily equal to the rate of increase of the total energy of the fluid; including (it must be emphasized) both the kinetic energy of part (b) and any thermal energy into which viscous dissipation may progressively convert that kinetic energy.

An estimate of the rate of increase of energy in part (b) may be derived from the rate, proportional to ρAU (where A is the body's frontal area), at which the mass of wake fluid is growing. Velocities in the vortex motion are proportional to U, giving a rate of increase of energy $\frac{1}{2}\rho AU^3C_D$, where C_D is a coefficient. The corresponding thrust required to yield this rate of working, and to overcome the equal and opposite vortex-flow drag of the fluid on the body, is

$$\tfrac{1}{2}\rho AU^2C_D. \tag{2}$$

Directly from the above results (1) and (2), for a body moving with variable velocity $U(t)$ in the negative x-direction through externally unbounded fluid at rest, we may obtain results for the important case of a body at rest in a flow of variable velocity $U(t)$ in the positive x-direction by the following simple device. We impose an additional velocity U, in the positive x-direction, on the whole system, requiring an additional uniform pressure gradient $-\rho\dot{U}$ in the fluid to generate the corresponding acceleration $\dot{U}$. These additional pressures, with gradient $-\rho\dot{U}$, produce an additional resultant force

$$+\rho\dot{U}V \tag{3}$$

on a body of volume V.

This force (3), sometimes known as the Froude–Krylov force, combines with the added-mass term (1) to give altogether a potential-flow component of drag

$$C_M\rho\dot{U}V, \tag{4}$$

where the Morison coefficient C_M is defined as

$$C_M = 1+(M_a/\rho V). \tag{5}$$

For example, C_M takes the value 1.5 for a sphere, or 2.0 for a circular cylinder.

Morison's equation expresses the total hydrodynamic drag D as a sum of its potential-flow component (4) and its vortex-flow component (2) in the form

$$D = C_M\rho\dot{U}V+\tfrac{1}{2}\rho AU^2C_D. \tag{6}$$

The rest of this paper is concerned with giving critical reviews both of the concept of analysing the hydrodynamic loading into two components in such a way and of matters concerning the magnitude and direction of action of both components; firstly in the idealized 'externally unbounded fluid' and then for real structures in real waves.

5. Basic principles concerning the momentum of a fluid flow

In the case of bodies in externally unbounded fluid we can obtain from momentum considerations expressions for hydrodynamic loading forces which, while maintaining a clear separation between potential-flow and vortex-flow forces, are at the same time more informative in two ways: they give vector forms of these forces that are not necessarily aligned with the velocity of the fluid, and they are quantitatively more precise. Nevertheless, there is a conceptual difficulty in applying momentum arguments to fluid flows.

This difficulty results from the classical ambiguity in how to define either the momentum of the disturbances to a flow of velocity (U_1, U_2, U_3) produced by a stationary body; or (what is the same thing) the momentum of the flow produced when the same body moves with velocity $(-U_1, -U_2, -U_3)$ in otherwise undisturbed fluid. This difficulty, resulting from the lack of absolute convergence in the integral defining momentum, can be surmounted in either of two equally acceptable ways, both well established in the literature of hydrodynamics.

One of these, expounded in detail by Lighthill (1979), defines the x_1-component of momentum as equal to its value for the fluid bounded internally by the body and externally by two parallel planes, both parallel to the x_1-axis. Note that the pressure force acting across those planes has no x_1-component. Therefore, the rate of change of the x_1-component of momentum as so defined is equal to the x_1-component of the force with which the body acts on the fluid.

This way of defining momentum by an integral which is absolutely convergent gives a result independent of which planes parallel to the x_1-axis are chosen provided that all of the vorticity in the flow field lies between them. Note, however, that a definition using planes not parallel to the x_1-axis would give a different value for the x_1-component of momentum; furthermore, that value would have no practical relevance because its rate of change would no longer balance the x_1-component of force with which the body acts on the fluid; differing from it, indeed, by an amount equal to the (in general) non-zero pressure force acting across those planes.

An alternative, more classical approach expounded rather comprehensively in Lamb's *Hydrodynamics*, and also developed a little further in Lighthill (1986), avoids referring to 'the momentum of the fluid' and refers rather to the impulse needed to set up a given motion from rest. For a changing flow field the rate of change of this impulse can be proved equal to the force with which the body acts on the fluid. Besides sharing this property with 'the momentum' as defined above, the value of the impulse is in all cases identical with that of the momentum.

The 'momentum' concept is the simpler one although this advantage is partly outweighed by the fact that a different definition is used for each component of momentum (clearly, for the x_2-component, we have to consider fluid bounded externally by parallel planes both parallel to the x_2-axis). The 'impulse' concept is, admittedly, simple for the potential-flow part of the fluid motion, which can always be set up by a distribution of impulses applied at its boundary, but involves more refined consideration for the vortex-flow part: the notional distribution of impulses needed to set this up includes impulses applied directly to particles of fluid throughout the rotational part of the flow. The total impulse, however, is (as stated earlier) identical with the total momentum. In the rest of this paper I use the word 'momentum' as a natural name for a quantity whose rate of change is equal to the applied force.

6. Momentum approaches to Morison's equation

For a general body shape without rotational symmetry its added mass for motion in different directions may take different values. However, there is necessarily a special set of coordinate axes (x_1, x_2, x_3) for which the added mass takes values M_{a1}, M_{a2}, M_{a3} and for which the potential-flow component (a) of the motion produced when the body moves with velocity $(-U_1, -U_2, -U_3)$ has momentum

$$(-M_{a1}\,U_1, -M_{a2}\,U_2, -M_{a3}\,U_3). \tag{7}$$

These special axes are called the body's three principal axes for fluid inertia.

It follows that the potential-flow component of the fluid motion acts on the body with a force

$$(M_{a1}\,\dot{U}_1, M_{a2}\,\dot{U}_2, M_{a3}\,\dot{U}_3) \tag{8}$$

equal and opposite to the force (given by the rate of change of the momentum (7)) with which the body acts upon that potential flow. When the body is at rest in an external flow with velocity (U_1, U_2, U_3) the potential-flow force acting on the body is the sum of (8) with the 'Froude–Krylov force'

$$\rho V(\dot{U}_1, \dot{U}_2, \dot{U}_3). \tag{9}$$

This sum can be written in terms of Morison coefficients

$$C_{M1} = 1 + (M_{a1}/\rho V), \quad \text{etc.} \tag{10}$$

as a force

$$(C_{M1}\rho\dot{U}_1\,V, C_{M2}\rho\dot{U}_2\,V, C_{M3}\rho\dot{U}_3\,V). \tag{11}$$

Here, the simple expression (4) for potential-flow force that occurs in Morison's equation is replaced by a more sophisticated and more accurate expression (familiar already to G. I. Taylor) that is not necessarily parallel to $(\dot{U}_1, \dot{U}_2, \dot{U}_3)$, the ambient fluid's acceleration vector – a vector which, of course, is in turn not necessarily parallel to that fluid's velocity vector (U_1, U_2, U_3). These are important refinements. On the other hand, the potential-flow force still has a linear dependence on the ambient velocity field, although this property (exact on the assumptions made above) will need to be reviewed critically for real structures in real waves.

G. I. Taylor clarified the fact (Taylor 1928*a*) that these potential-flow disturbances exhibit a dipole-like far-field behaviour. The disturbance potential is asymptotically equal to

$$V(C_{M1}\,U_1\,x_1 + C_{M2}\,U_2\,x_2 + C_{M3}\,U_3\,x_3)\,(x_1^2+x_2^2+x_3^2)^{-\frac{3}{2}} \tag{12}$$

as $(x_1^2+x_2^2+x_3^2)^{\frac{1}{2}}$ becomes large.

In addition to the contribution (7) of its potential-flow component (a), the total momentum of a fluid flow includes also the momentum of its vortex-flow component (b). A famous formula,

$$\tfrac{1}{2}\rho\int \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V, \tag{13}$$

for the momentum of a vortex flow is widely used for estimating the vortex-flow force on an offshore structure (as equal and opposite to the rate of change of this expression for the momentum). The expression (13) can be described in simple mechanical terms as one-half of the moment of a hypothetical force distribution with force per unit mass equal to the vorticity.

Like the potential-flow disturbances, the vortex motion (b) exhibits a dipole-like far-field behaviour. The vector dipole strength for its far-field behaviour is given by expression (13) divided by the density ρ.

Textbooks usually derive (13) for a fluid motion without solid boundaries. Where there is an internal boundary, however, (13) does still exactly represent the momentum of the vortex flow (*b*), satisfying zero boundary conditions, provided that $\boldsymbol{\omega}$ represents what is sometimes called the additional vorticity. This means the vorticity field minus a distribution of vorticity attached to the boundary in the form of a vortex sheet allowing exactly the tangential velocity (slip) associated with the potential flow (*a*).

For offshore structures, additional vorticity means, essentially, the shed vorticity. (It should on the other hand be noted that for flows around wings and winglike surfaces, where the local flow past an aerofoil section differs from simple potential flow by incorporating circulation around the aerofoil, additional vorticity includes bound vorticity as well as trailing vorticity.) In the language of §5, the fact that with $\boldsymbol{\omega}$ as additional vorticity the expression (13) represents the 'momentum' of the vortex flow (*b*) was demonstrated by Lighthill (1979); while Lighthill (1986) gives a thorough analysis of why the 'impulse' takes this form.

It follows that the simple scalar expression (2) for vortex-flow force that occurs in Morison's equation may (just as we found for the potential-flow term) be replaced by a more sophisticated and more accurate vector expression whose direction is not necessarily parallel to the ambient flow velocity. This expression is

$$-\frac{\mathrm{d}(\frac{1}{2}\rho\int \boldsymbol{x}\times\boldsymbol{\omega}\,\mathrm{d}V)}{\mathrm{d}t}, \tag{14}$$

being equal and opposite to the force with which the body is acting on the fluid to change the momentum (13) in the vortex-flow component (*b*) of the motion.

Equation (14) is valuable for structures in oscillating flows because it allows the vortex-flow force to be related directly to observations of vortex shedding and to understanding of how the shed vorticity is convected back and forth by a combination of the oscillatory potential flow (*a*) and of the vortex flow (*b*) itself. The entire hydrodynamic loading for externally unbounded flows may then be divided into

(*a*) the part (11) that depends linearly on the ambient velocity field and can be accurately calculated; and

(*b*) the part (14) that varies nonlinearly and can only be roughly estimated† but is related in a definite way to vortex shedding and to the convection of shed vorticity.

The rest of this paper is devoted to assessing the appropriateness of such a division for real structures in real waves and the extent to which the potential-flow force remains both linearly dependent and accurately calculable.

7. Potential-flow calculations with linearized free-surface condition

Among mathematical models that allow for the presence of the ocean's free surface, there is of course a very famous one under which the potential-flow force does remain linearly dependent and, with the aid of some excellent computer programs developed during the last ten years, accurately calculable. This is the model utilizing a linearized free-surface condition.

The propagation of ocean waves over relatively deep water has usually been seen as a particularly fruitful area of application of potential-flow theory. Indeed, in the

† We may, however, note that, although this paper's objectives do not include any survey of current numerical procedures for estimating vortex-flow forces, there have been impressive recent advances in the quantitative value of such procedures.

absence of any solid body, no source of vorticity is present so the flow can be expected to be irrotational.

The velocity potential ϕ then satisfies the linear Laplace equation together with a free-surface condition that can be approximately linearized, for waves of small steepness, in the form

$$\frac{\partial^2\phi}{\partial t^2} = -g\frac{\partial\phi}{\partial z} \quad \text{on } z = 0. \tag{15}$$

Here, z is a coordinate measured vertically upwards from the plane $z = 0$ representing the undisturbed position of the free surface.

Since on this model all the governing equations are linear, a general deformation of the sea surface can be Fourier-analysed into sinusoidal components with different wavenumbers k and different directions of propagation. For a sinusoidal wave propagating in the x-direction with amplitude a over deep water we have

$$\zeta_o = a \sin(\omega t - kx) \quad \text{and} \quad \phi_o = [k^{-1}\exp(kz)]\, a\omega \cos(\omega t - kx), \tag{16}$$

where ζ_o is the displacement of the free surface and ϕ_o the velocity potential, and where the frequency ω, by (15), satisfies

$$\omega^2 = gk. \tag{17}$$

The subscript o (for oncoming) is here used on ζ and on ϕ because the model proceeds to calculate the potential-flow force on an offshore structure generated by an oncoming wave specified as just the component (16) in the ambient sea-surface spectrum. A complete picture of the variation of potential-flow force on the structure can then be obtained from its combined response to all spectral components. We outline the classical wave-scattering theory for determining the potential-flow force on a structure in the oncoming wave (16) before discussing in §8 the legitimacy of its continued treatment separately from that of the vortex-flow force.

The linearized theory of the scattering of the oncoming wave (16) by a stationary structure expresses the linearized velocity potential (with l for linearized)

$$\phi_l = \phi_o + \phi_s \tag{18}$$

as a sum of oncoming and scattered waves. Here, ϕ_s like ϕ_o satisfies the linearized free-surface condition (15). Far from the structure, ϕ_s satisfies 'the radiation condition' stating that it represents only outgoing waves. At the surface of the structure, ϕ_s satisfies the condition

$$\frac{\partial\phi_s}{\partial n} = -\frac{\partial\phi_o}{\partial n}, \tag{19}$$

stating that the combined velocity potential (18) has its local value of normal velocity $\partial\phi/\partial n$ equal to zero. To the same linearized approximation, the potential-flow force on the structure is the resultant action

$$\int_{S_-} \rho(\partial\phi_l/\partial t)\,\boldsymbol{n}\,\mathrm{d}S \tag{20}$$

on S_- (part of the structure's surface below the undisturbed water level $z = 0$) of the transient pressure $-\rho\,\partial\phi_l/\partial t$ (linearized form of the pressure excess over its hydrostatic distribution).

Good computer programs exist (see, for example, Hogben *et al.* 1977; Standing 1981) for determining ϕ_s, or at least for determining its distribution over S_- which is required in the calculation of the potential-flow force (20). Extensive information is available, therefore, on how this force varies as a function of wavenumber for different shapes of structure.

A feature common to all of the data so obtained is their limiting behaviour for low wavenumber. Essentially, the potential-flow force in this limiting case is given to close approximation by the methods of §6 above. Specifically, any piece of the structure with cross-sectional dimensions small compared to k^{-1} (that is, to $\lambda/2\pi$, where λ is the wavelength) responds as in (11) to the fluctuating ambient velocity (U_1, U_2, U_3) in its immediate neighbourhood. This velocity may be taken as the value of grad ϕ_0 at (say) the centroid of the piece of structure because its size is given as small compared with the scale k^{-1} of variation of expression (16) for ϕ_0.

Thus the computer programs in question have twofold importance. First of all, they confirm the validity of the methods of §6 for application to the vast majority of structural elements; namely, those with cross-sectional dimensions significantly smaller than a typical ocean wavelength divided by 2π. Secondly, they allow accurate results to be obtained for that relatively small number of elements within offshore structures (large storage tanks, for example) that do not satisfy this condition.

8. Effects of free surface on vortex-flow forces

The practical value of all the above conclusions might, on the other hand, be questioned on the basis of doubts regarding the validity, when a free surface is present, of the division of hydrodynamic loadings into (*a*) potential-flow forces and (*b*) vortex-flow forces. These areas of possible uncertainty may, however, be explored fruitfully from the viewpoint of the analysis given in §6.

This analysis indicates, indeed, that important common features are shared by potential-flow force and vortex-flow force. In particular, both are intimately linked to the vector dipole strength for an associated far-field behaviour. Indeed, if this dipole strength is written as $\boldsymbol{G}$, then the associated force, given in the two cases by expressions (11) and (14), can in either case be written as

$$-\rho\dot{\boldsymbol{G}}. \tag{21}$$

The relevance of this fact is most evident in the limit of low wavenumber (§7). Then the verification that the local dipole-like disturbance to the potential flow, produced by a piece of the structure small compared with k^{-1}, generates to close approximation the same force as in externally unbounded fluid, can be used to draw a similar inference for the local dipole-like vortex-flow disturbance. This inference is that vorticity extending over a region small compared with k^{-1} will generate a force given by the same expression (14) as in externally unbounded fluid.

Consider, for example, a plane vortex ring, with circulation K around the core of that line vortex whose looped shape constitutes the ring. In this case, the associated flow field could be described by a velocity potential ϕ only if we allowed the value of ϕ at a point to increase by K as that point encircled the line vortex. Therefore, the motion could be described by a single-valued potential ϕ only if a discontinuity were allowed; for example, a discontinuous jump by K at the plane surface having the ring as its boundary. This means that the vortex ring is exactly equivalent to a distribution over this plane surface of dipoles whose vector strength per unit area has magnitude K and direction normal to this plane surface. The far-field behaviour is therefore that of a dipole of strength

$$\boldsymbol{G} = \frac{1}{2}\int \boldsymbol{x} \times \boldsymbol{\omega}\,\mathrm{d}V, \tag{22}$$

(of magnitude K times the area enclosed by the vortex ring and the same direction) exactly as stated in §6.

This classically familiar argument has been summarized here (see, for example, Lighthill (1986) for a fuller treatment) simply in order to remind the reader that it remains exactly valid whether or not the fluid has a free surface. In the presence of a free surface where the boundary condition (15) is satisfied, the dipole has of course a mathematically more complicated velocity field. Nevertheless, in those low-wavenumber conditions when the potential-flow force has the same relation to dipole strength as for unbounded fluid, we can expect the same to be the case for the vortex-flow force.

This conclusion for a vortex ring can then be extended to a more complicated vorticity distribution by regarding it as composed of an ensemble of such rings. It suggests that treatment of the vortex-flow force (*b*) in the manner outlined in §6 may be appropriate in low-wavenumber cases. In other cases, admittedly, no such conclusion can be drawn, but these are cases when vortex-flow forces are expected according to the estimate given by Morison's equation (6) to be small compared with potential-flow forces. We infer that the careful calculation of potential-flow forces is important under all circumstances.

9. Nonlinear theories of potential-flow force

Lighthill (1979) suggested a number of reasons for dissatisfaction, even in low-wavenumber cases, with expressions for potential-flow force having a linear dependence on the ambient velocity field. His work together with similar work by others at that time (see Mei (1983) for a detailed survey of the literature) demonstrated that, in all cases, reliable calculations of potential-flow force to second order could be made, and that the calculated contributions having quadratic dependence on wave amplitudes could be of substantial magnitude. Because the approximate estimation of vortex-flow forces through the use of expressions like (14) requires particularly refined investigation, it is all the more important that any resulting estimates of vortex-flow forces should be able to be compared with experimental measurements from which potential-flow forces, calculated as accurately as possible (namely, to second order), have been subtracted.

Lighthill (1979) contrasted the above approach with an alternative, long established procedure for taking effects of large wave steepnesses into account. The latter procedure is based on the consideration of an oncoming wave in the form of an exactly calculated periodic wave of large steepness, whose flow field is then used as the ambient velocity (U_1, U_2, U_3) when flow-force formulae such as those of §6 are applied. In particular, the potential-flow force is estimated from the linear formula (11) applied to the velocities in the well-known 'Stokes wave' (periodic wave satisfying a full nonlinear free-surface condition).

Such a procedure is logically unsatisfactory in many different ways. First, because Fourier analysis is no longer available when a nonlinear free-surface condition is used, there is no compelling argument for considering in isolation the case of a periodic wave. Secondly, the procedure takes nonlinear effects into account only as they influence the oncoming wave while neglecting nonlinear effects on the interaction between that wave and the structure. Yet, as demonstrated by Lighthill (1979) and others, this interaction is influenced by nonlinear effects to a greater extent than is the undisturbed wave. In fact, substantial contributions of second order in the wave amplitude to the potential-flow force resulting from this interaction can in practice be calculated (as already mentioned); while, for the free wave, nonlinear effects arise only at third order.

From an even more fundamental standpoint, the exclusive use of expression (11) to derive the potential-flow force with which a 'Stokes wave' acts upon a structural component is unsatisfactory because it represents the ambient flow around that component in terms only of the local velocity (U_1, U_2, U_3) of that flow. It was G. I. Taylor himself, however (Taylor 1928*b*), who emphasized that the ambient rate-of-strain (part (*a*) of the analysis of the local motion given in §2 above) generates in potential flow a significant interaction when a solid body is present to oppose that straining motion. Now, Lighthill (1979) pointed out that the sinusoidal-wave potential ϕ_0 of (16) generates large extensional rate-of-strain components

$$\frac{\partial^2 \phi_0}{\partial x^2} = -\frac{\partial^2 \phi_0}{\partial z^2} = -[k \exp(kz)]\, a\omega \cos(\omega t - kx), \tag{23}$$

and confirmed that the presence of a body in this fluctuating rate-of-strain field produces an important potential-flow response.

He showed furthermore that the second-order force on the body arises from cross terms between its responses to fluctuating velocity and to fluctuating rate-of-strain rather than from squared terms in either response separately. This agrees with the conclusion of Taylor (1928*b*), who in the notation of §6 above identified these cross-terms for a body in externally unbounded fluid as a force of which the x_j-component is

$$\rho V\left(C_{M1}\, U_1 \frac{\partial U_1}{\partial x_j} + C_{M2}\, U_2 \frac{\partial U_2}{\partial x_j} + C_{M3}\, U_3 \frac{\partial U_3}{\partial x_j}\right). \tag{24}$$

Lighthill's corresponding result for flows with a free surface includes three terms, one of which reduces to expression (24) in the low-wavenumber limit.

10. Loading associated with the nonlinear free-surface condition

Another of Lighthill's three terms represents the effect of refining the free-surface condition (15) so that it becomes accurate to the second order in small quantities. To this higher approximation it takes the form

$$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = -\frac{\partial (\nabla \phi)^2}{\partial t} \quad \text{on } z = \zeta. \tag{25}$$

Amusingly, the right-hand side is double the value derived by simply taking into account the dynamic pressure term

$$-\tfrac{1}{2}\rho(\nabla\phi)^2, \tag{26}$$

when equating water pressure to atmospheric pressure at the free surface; this is because an exactly equal contribution appears (Lighthill 1979, figure 24) when the rate of change of the sum of transient and hydrostatic pressure on the free surface is expressed to second order.

The simple periodic oncoming wave on deep water, given by (16), satisfies the linearized free-surface condition (15) not just on $z = 0$ but for all values of z. Furthermore, $(\nabla\phi_0)^2$ takes the value $a^2\omega^2 \exp(2kz)$, independent of t, so that the second-order condition (25) is also satisfied for all z and, in particular, on $z = \zeta$. In short, the free sinusoidal wave (as mentioned in §9) is unaltered by second-order effects.

Its interaction with the structure, however, written in (18) as ϕ_1, is altered in two ways by second-order effects. If we expand the velocity potential of the irrotational part of the flow as

$$\phi = \phi_1 + \phi_q + \ldots, \tag{27}$$

where ϕ_1 satisfies the linearized free-surface condition (15) and ϕ_q is the quadratic (that is, second-order) correction to ϕ_1, then the free-surface condition on ϕ_q becomes

$$\frac{\partial^2 \phi_q}{\partial t^2} + \frac{g\,\partial \phi_q}{\partial z} = -\frac{\partial (\nabla \phi_1)^2}{\partial t} - L \quad \text{on } z = 0. \tag{28}$$

Here, the second term on the right-hand side, absent for the particular case considered by Lighthill (1979) (namely a vertical cylinder, for which as for the free sinusoidal wave the linearized condition (15) is satisfied for all z), takes in general the form

$$L = \zeta_1 \left[\frac{\partial\,(\partial^2 \phi_1/\partial t^2 + g\,\partial \phi_1/\partial z)}{\partial z} \right]_{z=0}. \tag{29}$$

This effect, due to the vertical gradient of the left-hand side of (25), supplements in (28) the effect of the first term due to variability of free-surface speed in the wave-structure interaction potential ϕ_1.

The condition (28) essentially equates ϕ_q to the solution of a linear problem of wave radiation resulting from the action of a fluctuating pressure Q applied at the free surface in the presence of the stationary structure. Here, Q fluctuates at double the frequency of the oncoming wave. In problems (including the case of the vertical cylinder) where $L = 0$, the applied surface pressure Q takes the value $\rho(\nabla\phi_1)^2$; in more general cases,

$$\rho^{-1} \frac{\partial Q}{\partial t} = \frac{\partial (\nabla \phi_1)^2}{\partial t} + L. \tag{30}$$

Lighthill shows how easy it is to determine any particular component of the additional loading on the structure which results from the quadratic potential ϕ_q. This determination can be regarded as a straightforward application of the general reciprocity principle for mechanical systems. This principle refers to two alternative sets of displacements, generated by two alternative sets of forces, and states that the work done by the forces of either set acting over the displacements of the other is the same.

We apply this principle taking as the first set of forces those that are needed to generate the potential ϕ_q. These include the applied surface pressures Q and also forces needed to keep the structure stationary (these are forces equal and opposite to the hydrodynamic loadings on the structure).

Suppose now that we need to know the resultant F_q of the hydrodynamic loadings in any particular direction. Then we take the second set of forces to be those which would be required to give the structure unit translational motion in that direction; specifically, a motion fluctuating at the same frequency (twice the natural frequency of the oncoming wave) with unit velocity amplitude. This motion radiates waves on the free surface (which we suppose to be completely free in this second case) and the distribution W of vertical velocity over the free surface can in principle be computed for this second linear problem.

The rate of working by the forces of the first set acting over the displacements of the second set is

$$-F_q - \int_{z=0} WQ\,\mathrm{d}S. \tag{31}$$

This results from the force $(-F_q)$ acting in the direction of the translational motion with unit velocity, and from the force $Q\,\mathrm{d}S$ exerted by the applied pressures in the negative z-direction, opposite to that of the vertical velocity W. By contrast, the rate of working by the forces of the second set (applied entirely at the structure) on the displacements of the first set (with the structure at rest) is zero. The reciprocity principle equates (31) to zero, then, giving

$$F_q = -\int_{z=0} WQ\,\mathrm{d}S. \tag{32}$$

Equation (32) exemplifies ways of calculating second-order forces from quantities computed by solving linear problems. The same method is easily adapted to give, for example, the moment of the loading about a particular axis (the second set of forces is then taken to be those needed to give the structure unit rotational motion about that axis).

Eatock Taylor & Hung (1986) give a most valuable review of procedures for obtaining F_q. Their paper derives results for any wavenumber k, with and without the above restriction to the deep-water case.

11. Additional second-order loadings

Besides the loading F_q that is associated with the quadratic potential ϕ_q (but calculable as we have seen from the solutions to linear problems), there are two parts of the second-order hydrodynamic loading which result directly from the linear potential ϕ_1. One of these arises from the fact that the loading calculated on linear theory is given by an integral (20) over just that part of the structure's surface, S_-, that lies below the undisturbed water level $z = 0$. If however that surface intersects the plane $z = 0$ in a waterline w, then a certain additional 'waterline force',

$$\int_w \tfrac{1}{2}\rho g \zeta_1^2(-\boldsymbol{n})\,\mathrm{d}s = \tfrac{1}{2}\rho g^{-1}\int_w \left(\frac{\partial\phi_1}{\partial t}\right)^2(-\boldsymbol{n})\,\mathrm{d}s, \tag{33}$$

acts at the waterline; resulting from linear-theory pressures either, where $\zeta_1 > 0$, acting on a part of the structure above $z = 0$ or, where $\zeta_1 < 0$, failing to act on a part of the structure below $z = 0$ (see Lighthill 1979, figure 27). Here, $\mathrm{d}s$ is an element of length of the plane curve w constituting the waterline and $\boldsymbol{n}$ is the outward normal to that curve.

Finally, the linear-theory potential ϕ_1 generates directly one further second-order force, associated with the dynamic pressures $-\frac{1}{2}\rho(\boldsymbol{\nabla}\phi_1)^2$, which necessarily accompany the linear-theory pressures (transient plus hydrostatic) just referred to. Their resultant may be named the 'dynamic force',

$$\int_{S_-} \tfrac{1}{2}\rho(\boldsymbol{\nabla}\phi_1)^2\,\boldsymbol{n}\,\mathrm{d}S. \tag{34}$$

The total second-order force is the sum of three parts given by expressions (32), (33) and (34). All are directly calculable in terms of computed solutions to linear wave-radiation problems. Lighthill (1979) draws attention to the value of this additional benefit derivable from such computed solutions. They allow the calculation of the potential-flow force not just to a linearized approximation but also to second order. This permits the determination of vortex-flow force (by subtraction of the calculated potential-flow force from measured data) to greatly improved accuracy;

which in turn facilitates its convincing interpretation in terms of the convection of shed vorticity, in accordance with (14).

In low-wavenumber cases, the dynamic force (34) on each component of the structure coincides with the value (24) obtained for externally unbounded fluid by the method of G. I. Taylor. This value, as noted earlier, takes the form of cross-terms between the component's response to ambient velocity and its response to ambient rate-of-strain. Such a form is, furthermore, taken (Lighthill 1979) by all three parts of the second-order force in the low-wavenumber limit; a limiting process in which the waterline force and the dynamic force are comparable in magnitude.

For small enough wavenumber k, on the other hand, the quadratic force becomes negligible by comparison with the dynamic and waterline forces (Lighthill 1979).† Therefore, the sum of the waterline force (33) and a dynamic force taking Taylor's form (24) gives to good approximation the appropriate second-order correction, for structural components with cross-section small compared with k^{-1}, to the simple potential-flow forces represented in this limit by the expression (11) using Morison coefficients. Lighthill (1979) made clear the quantitative significance of this conclusion for the response of a structure to a periodic wave.

12. Application to structures with very low natural frequencies

In this paper I have outlined those directions of research that first suggested how second-order potential-flow forces may need to be taken seriously in the interactions between waves and structures. In later developments a most important application of this theoretical approach has proved to be to the new types of oil platform, designed for operation in deeper areas of the continental shelf, such as the tension-leg platform introduced by Conoco in Britain's Hutton Field. These are relatively compliant structures, with very low natural frequencies. It might indeed be claimed for them that their natural frequencies are so low (corresponding to periods of the order of a minute) that no ocean waves are directly able to excite them. The practical significance of such a claim is evident.

On the other hand, the second-order potential-flow forces that are generated by an ocean-wave spectrum include difference-tone components, related to that part of the product of two nearby components in the ocean-wave spectrum whose frequency, given by the difference of their two frequencies, resonates with the very low natural frequency of the compliant structure. Second-order forces at such difference frequencies are found to be magnified in importance, simply because they are able to generate a resonant response in the structure.

Several investigators have studied this phenomenon (e.g. Newman 1975; Pinkster 1979; Standing, Dacunha & Matten 1981). In a major programme of current research further light is being shed on the second-order forces, their sensitivity to hydrodynamic interactions in complex compliant structures, and their implications for low-frequency responses (Drake, Eatock Taylor & Matsui 1984; Eatock Taylor & Hung 1985; Eatock Taylor & Sincock 1986; Matsui 1986). The broad principles outlined in this paper are being found to have particular value in this context.

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† More recently, however, Eatock Taylor & Hung (1986), while confirming Lighthill's results for low wavenumber k, showed how the quadratic force (32) grows rapidly as k increases.

REFERENCES

DRAKE, K. R., EATOCK TAYLOR, R. & MATSUI, T. 1984 Drift of an articulated cylinder in regular waves. *Proc. R. Soc. Lond.* A **394**, 363–385.

EATOCK TAYLOR, R. & HUNG, S. M. 1985 Wave drift enhancement effects in multi-column structures. *Appl. Ocean Res.* **7**, 128–139.

EATOCK TAYLOR, R. & HUNG, S. M. 1986 Second order diffraction forces on a vertical cylinder in regular waves. *Appl. Ocean Res.* (in press).

EATOCK TAYLOR, R. & SINCOCK, P. 1986 Comparison of analyses for moored systems in random waves. *Proc. Fifth Offshore Mechanics and Arctic Engng Symp.*, vol. 1, pp. 264–272. American Society of Mechanical Engineers, New York, USA.

HOGBEN, N., MILLER, B. L., SEARLE, J. W. & WARD, G. 1977 Estimation of fluid loading on offshore structures. *Natl Mar. Inst. Rep.* no. 11.

LIGHTHILL, J. 1963 *Laminar Boundary Layers* (ed. L. Rosenhead), chapters 1 and 2. Oxford University Press.

LIGHTHILL, J. 1979 Waves and hydrodynamic loading. *Proc. 2nd Int. Conf. Behaviour of Off-Shore Structures*, vol. 1, pp. 1–40. BHRA Fluid Engineering, Cranfield, Bedford, England.

LIGHTHILL, J. 1986 *An Informal Introduction to Theoretical Fluid Dynamics.* Oxford University Press.

MATSUI, T. 1986 Analysis of slowly varying wave drift forces on compliant structures. *Proc. Fifth Offshore Mechanics and Arctic Engng Symp.*, vol. 1, pp. 289–296. American Society of Mechanical Engineers, New York, USA.

MEI, C. C. 1983 *The Applied Dynamics of Ocean Surface Waves.* Wiley.

NEWMAN, J. N. 1975 Second order slowly varying forces on vessels in irregular waves. *Proc. Int. Symp. Dynamics of Marine Vehicles and Structures in Waves*, pp. 182–186. Institute of Mechanical Engineers, London.

PINKSTER, J. A. 1979 Mean and low frequency wave drifting forces on floating structures. *Ocean Engng* **6**, 593–615.

STANDING, R. G. 1981 Wave loading on offshore structures. *Natl Mar. Inst. Rep.* no. R102.

STANDING, R. G., DACUNHA, N. M. C. & MATTEN, R. B. 1981 Slowly varying second order wave forces: theory and experiment. *Natl Mar. Inst. Rep.* no. R138.

TAYLOR, G. I. 1928*a* The energy of a body moving in an infinite fluid, with an application to airships. *Proc. R. Soc. Lond.* A **120**, 13–21.

TAYLOR, G. I. 1928*b* The forces on a body placed in a curved or converging stream of fluid. *Proc. R. Soc. Lond.* A **120**, 260–283.